Complex Zeros; Fundamental Theorem of Algebra

Complex Zero: A complex zero is in the form $a+bi$ or $a-bi$.

Fundamental Theorem of Algebra: Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form $f(x) = a_n(x-r_1)(x-r_2)\ldots(x-r_n)$, where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Conjugate Pairs Theorem: For any complex polynomial function, if $r = a+bi$ is a zero of f, the complex conjugate $\bar{r} = a-bi$ is also a zero of f. ★ Imaginary solutions always come in pairs.

Corollary: A polynomial f of odd degree with real coefficients has at least one real zero.

Example: A polynomial f of degree 5 has the zeros 2, 3*i*, and 4 + *i*. Find the other two zeros.

$-3i$ & $4-i$

Conjugates of $3i$ & $4+i$

Example: Find a polynomial f of degree 4 that has the zeros 5, –3, and –2 + *i*. Additional zero: $-2-i$

$f(x) = (x-5)(x+3)(x-(-2+i))(x-(-2-i))$

$f(x) = (x^2-2x-15)(x+2-i)(x+2+i)$

$f(x) = (x^2-2x-15)(x^2+4x+5)$

$\boxed{f(x) = x^4+x^3-18x^2-70x-75}$

	x	$+2$	$-i$
x	x^2	$+2x$	$\cancel{-ix}$
$+2$	$+2x$	$+4$	$\cancel{-2i}$
$+i$	$\cancel{+ix}$	$\cancel{+2i}$	$-i^2 = +1$

$(x+2-i)(x+2+i)$
$= x^2+4x+5$

	x^2	$-2x$	-15
x^2	x^4	$-2x^3$	$-15x^2$
$+4x$	$+4x^3$	$-8x^2$	$-60x$
$+5$	$+5x^2$	$-10x$	-75

$(x^2-2x-15)(x^2+4x+5)$
$= x^4+2x^3-18x^2-70x+75$

Example: Find a polynomial f of degree 5 with the following zeros: 1, *i*, 4 – 2*i*. Other zeros: $-i$, $4+2i$

$f(x) = (x-1)(x-i)(x+i)(x-(4-2i))(x-(4+2i))$

$f(x) = (x-1)(x^2+ix-ix\overset{+1}{\cancel{-i^2}})(x-4+2i)(x-4-2i)$

$f(x) = (x-1)(x^2+1)(x-4+2i)(x-4-2i)$

$f(x) = (x^3-x^2+x-1)(x^2-8x+20)$

$\boxed{f(x) = x^5-9x^4+29x^3-29x^2+28x-20}$

	x	-4	$+2i$
x	x^2	$-4x$	$\cancel{+2ix}$
-4	$-4x$	$+16$	$\cancel{-8i}$
$-2i$	$\cancel{-2ix}$	$\cancel{+8i}$	$-4i^2 = +4$

$(x-4+2i)(x-4-2i)$
$= x^2-8x+20$

	x^3	$-x^2$	$+x$	-1
x^2	x^5	$-x^4$	$+x^3$	$-x^2$
$-8x$	$-8x^4$	$+8x^3$	$-8x^2$	$+8x$
$+20$	$+20x^3$	$-20x^2$	$+20x$	-20

$(x^3-x^2+x-1)(x^2-8x+20)$
$= x^5-9x^4+29x^3-29x^2+28x-20$

Example: Find the complex zeros of $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$. Write f in factored form.

4 zeros

Factors of 65: $\pm 1, \pm 5, \pm 13, \pm 65$

Factors of 2: $\pm 1, \pm 2$

Potential zeros: $\pm 1, \pm 5, \pm 13, \pm 65, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{13}{2}, \pm \frac{65}{2}$

$f(1) = -80$ $\quad f(-1) = 144$ $\quad \underline{f(5) = 0}$ $\quad f(-5) = 880$ $\quad f(13) = 52{,}000$ $\quad \underline{f(\frac{1}{2}) = 0}$

$$\begin{array}{r|rrrrr} 5 & 2 & 1 & -35 & -113 & 65 \\ & & 10 & 55 & 100 & -65 \\ \hline & 2 & 11 & 20 & -13 & \multicolumn{1}{|r}{0} \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 11 & 20 & -13 \\ & & 1 & 6 & 13 \\ \hline & 2 & 12 & 26 & \multicolumn{1}{|r}{0} \end{array}$$

$f(x) = (x-5)(x-\frac{1}{2})(2x^2+12x+26)$

$f(x) = 2(x-5)(x-\frac{1}{2})(x^2+6x+13)$

use quadratic formula

$x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \dfrac{-6 \pm \sqrt{-16}}{2} = \dfrac{-6 \pm 4i}{2} = -3 \pm 2i$

$f(x) = 2(x-5)(x-\frac{1}{2})(x-(-3+2i))(x-(-3-2i))$

$$\boxed{\begin{array}{l} f(x) = 2(x-5)(x-\frac{1}{2})(x+3-2i)(x+3+2i) \\ \text{zeros: } 5, \frac{1}{2}, -3+2i, -3-2i \end{array}}$$

1	252	9	28
2	126	12	21
3	84	14	18
4	63		
6	42		
7	36		

Example: Find the complex zeros of $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$. Write f in factored form.

4 zeros

Factors of 252/potential zeros:

$\pm 1, \pm 252, \pm 2, \pm 126, \pm 3, \pm 84, \pm 4, \pm 63, \pm 6, \pm 42, \pm 7, \pm 36, \pm 9, \pm 28, \pm 12, \pm 21, \pm 14, \pm 18$

$f(-1) = -300$ $\quad f(-2) = -390$ $\quad f(-3) = -504$

$f(-4) = -600$ $\quad f(-6) = -450$ $\quad \underline{f(-7) = 0}$

$$\begin{array}{r|rrrrr} -7 & 1 & 3 & -19 & 27 & -252 \\ & & -7 & 28 & -63 & 252 \\ \hline & 1 & -4 & 9 & -36 & \multicolumn{1}{|r}{0} \end{array}$$

$f(x) = (x+7)(x^3 - 4x^2 + 9x - 36)$

can factor by grouping

$f(x) = (x+7)[x^2(x-4) + 9(x-4)]$

$f(x) = (x+7)(x-4)(x^2+9)$

$x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9} = \pm 3i$

$$\boxed{\begin{array}{l} f(x) = (x+7)(x-4)(x+3i)(x-3i) \\ \text{zeros: } -7, 4, -3i, 3i \end{array}}$$

Example: One of the zeros of $f(x) = x^4 - 2x^3 - 3x^2 + 10x - 10$ is $1+i$. Find the remaining zeros.

Additional zero: $1-i$

$(x-(1+i))$ & $(x-(1-i))$ are factors

$(x-1-i)(x-1+i) = x^2-2x+2$

x^2-2x+2 is a factor

	x	-1	$-i$
x	x^2	$-x$	$\cancel{-ix}$
-1	$-x$	$+1$	$\cancel{+i}$
$+i$	$\cancel{+ix}$	$\cancel{-i}$	$-i^2 = +1$

$$
\begin{array}{r|l}
 & x^2 - 5 \\
\hline
x^2-2x+2 & x^4 - 2x^3 - 3x^2 + 10x - 10 \\
 & -(x^4 - 2x^3 + 2x^2) \\
\hline
 & -5x^2 + 10x - 10 \\
 & -(-5x^2 + 10x - 10) \\
\hline
 & 0
\end{array}
$$

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5}$

$f(x) = (x^2-2x+2)(x^2-5)$

$f(x) = (x-1-i)(x-1+i)(x+\sqrt{5})(x-\sqrt{5})$

$\boxed{\text{zeros: } 1+i,\ 1-i,\ -\sqrt{5},\ \sqrt{5}}$